Solutions to Assignment 2

1. Let $C^\infty_{2\pi}$ be the class of all smooth 2π-periodic, complex-valued functions and $\mathcal{C}^\infty$ the class of all complex bisequences satisfying $c_n = \circ(n^{-k})$ as $n \to \pm\infty$ for every k. Show that the Fourier transform $f \mapsto \hat{f}$ is bijective from $C^\infty_{2\pi}$ to $\mathcal{C}^\infty$.

   Solution First, we show that the Fourier coefficients of a smooth, periodic function are rapidly decreasing. A repeated application of Problem 1 shows that $(in)^k \hat{f}(n)$ is equal to the Fourier coefficients of $f^{(k)}$ for every k. In general, we have

$$\begin{aligned} |\hat{g}(n)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\{\pi} g(x)e^{-inx}\,dx\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|g(x)||e^{-inx}|dx \\ &\le \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}|g(x)|dx\right| \equiv M(g)\ , \end{aligned}$$

   that is, the Fourier coefficients of any integrable function are always uniformly bounded. Now, for a fixed k, we have

$$|\hat{f}(n)| = \left|\frac{1}{(in)^k}\hat{f^{(k)}}(n)\right| \le \frac{M(f^{(k)})}{n^k}\ ,$$

   so $\{\hat{f}(n)\}$ belongs to $\mathcal{C}^\infty$.

   Second, onto. Let $\{c_n\}$ be a rapidly decreasing bisequence. Define

$$f(x) \equiv \sum_{-\infty}^{\infty} c_n e^{inx}\ .$$

   Taking $k = 2$, we have

$$\left|c_n e^{inx}\right| = |c_n| \le \frac{C}{n^2},$$

   for some constant C. By M-Test the right hand side in f is a uniformly convergent series of functions so f is well-defined. Furthermore, as uniform convergence preserves continuity, f is also continuous. By using M-Test to $\sum_{-\infty}^{\infty} inc_n e^{inx}$ (taking $k = 3$), we see that it is also uniformly convergent. By one exchange theorem we learned in 2060 we conclude that f is differentiable and $f'(x) = \sum_{-\infty}^{\infty} inc_n e^{inx}$. Repeating this argument we see that $f \in C^\infty_{2\pi}$.

   Third, one-to-one. By Theorem 1.7 $f(x) = \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx}$ and $g(x) = \sum_{-\infty}^{\infty} \hat{g}(n)e^{inx}$. When $\hat{f}(n) = \hat{g}(n)$, it is obvious that $f \equiv g$.

2. Propose a definition for $\sqrt{d/dx}$. This operator should be a linear map which maps $C^\infty_{2\pi}$ to itself satisfying

$$\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}}f = \frac{d}{dx}f,$$

   for all smooth, 2π-periodic f.

   Solution Use complex notation. For a smooth function f,

$$\widehat{f'}(n) = in\hat{f}(n). \tag{1}$$

In view of $i = e^{i\pi/2}$, this motivates us to define $g(x) = \sqrt{d/dx} f(x)$ to be the function whose Fourier series is given by

$$\hat{g}(n) = c_n = e^{i\pi/4}\sqrt{n}\hat{f}(n).$$

That is,

$$g(x) = \sum_{n=-\infty}^{\infty} e^{i\pi/4}\sqrt{n}\hat{f}(n)e^{inx} \ .$$

When $f \in C^\infty_{2\pi}$, by Problem 5 in Assignment 1 (see also the previous problem), it is easy to see that the series in the right hand side of g defines again a smooth and 2π-periodic function, and the convergence is uniform. Hence $\sqrt{d/dx}$ is a linear map on $C^\infty_{2\pi}$ to itself. Writing $h(x) = \sqrt{\dfrac{d}{dx}}\sqrt{\dfrac{d}{dx}} f(x)$, then

$$\hat{h}(n) = e^{i\pi/4}\sqrt{n}\hat{g}(n) = e^{i\pi/4}\sqrt{n}e^{i\pi/4}\sqrt{n}\hat{f}(n) = (in)\hat{f}(n).$$

By the the uniqueness of the Fourier series, one has

$$\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}} f = \frac{d}{dx} f.$$

This problem demonstrates the power of Fourier series. It is hopeless to define fractional derivative on the function directly.

3. Let f be a continuous, 2π-periodic function and its primitive function be given by

$$F(x) = \int_0^x f(x)dx.$$

Show that F is 2π-periodic if and only if f has zero mean. In this case,

$$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \quad \forall n \neq 0.$$

Solution. From

$$\begin{aligned} F(x+2\pi) &= \int_0^{x+2\pi} f(y)\ dy \\ &= \int_0^{2\pi} f(y)\ dy + \int_{2\pi}^{x+2\pi} f(y)\ dy \\ &= \int_0^{2\pi} f(y)\ dy + \int_0^{x} f(y)\ dy \\ &= \int_0^{2\pi} f(y)\ dy + F(x)\ , \end{aligned}$$

it is clear that F is of period 2π if and only if f has zero mean. The formula comes by easily.

4. Let $\mathcal{C}'$ be the subspace of $\mathcal{C}$ consisting of all bisequences $\{c_n\}$ satisfying $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$.

   (a) For $f \in R[-\pi, \pi]$, show that

$$2\pi \sum_{-\infty}^{\infty} |c_n|^2 \leq \int_{-\pi}^{\pi} |f|^2\ .$$

(b) Deduce from (a) that the Fourier transform $f \mapsto \hat{f}(n)$ maps $R_{2\pi}$ into $\mathcal{C}'$.

(c) Explain why the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha}\ , \quad \alpha \in (0, 1/2]\ ,$$

is not the Fourier series of any function in $R_{2\pi}$.

Solution. (a) Using $\left(f(x) - \sum_{k=-n}^{n} c_k e^{ikx}\right) \overline{\left(f(x) - \sum_{k=-n}^{n} c_k e^{ikx}\right)} \geq 0$ for all n and x,

$$\begin{aligned}
0 \ &\leq \ \int (f(x) - \sum_{k=-n}^{n} c_k e^{ikx}) \overline{(f(x) - \sum_{k=-n}^{n} c_k e^{ikx})}\, dx \\
&= \ \int (f(x) - \sum_{k=-n}^{n} c_k e^{-ikx})(\overline{f(x)} - \sum_{j=-n}^{n} \overline{c_j} e^{-ijx})\, dx \\
&= \ \int (|f(x)|^2 - \sum_{j=-n}^{n} f(x)\overline{c_j} e^{-ijx} - \sum_{k=-n}^{n} \overline{f(x)} c_k e^{ikx} + \sum_{j,k=-n}^{n} c_j \overline{c_k} e^{i(j-k)x})\, dx \\
&= \ \int (|f(x)|^2 - 2\pi \sum_{k=-n}^{n} |c_k|^2)\, dx\ ,
\end{aligned}$$

by the orthogonality of e^{-ikx}'s. The desired inequality follows by letting n go to infinity.

(b) It is clear from (a).

(c) From $\sum |c_n|^2 < \infty$ one deduces that $\sum a_n^2, \sum b_n^2 < \infty$ also hold when the function is of real-valued. Now, if the given trigonometric series come from an integrable function, then $\sum a_n^2 = \sum \dfrac{1}{n^{2\alpha}}$ must be finite. But now it is not when $\alpha \in (0, 1]$. We conclude that it is not a Fourier series.